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13. ABSTRACT *(Maximum 200 words)*

During this one-year project, we continued our research on the development, analysis, and application of serial and parallel adaptive computational strategies for solving transient and steady partial differential systems. We concentrated on high-order methods and adaptive approaches that unite mesh refinement and coarsening (h-refinement), order variation (p-refinement), and mesh motion (r-refinement). Parallel computational techniques involved load-balancing and load-redistribution strategies for implementing these adaptive methods on distributed-memory MIMD computers. In particular, we have developed migration strategies that exchange finite elements between neighboring spatial domains of different processors. Effective load balancing in an adaptive setting requires speedy procedures since balancing must be performed frequently. Migration offers several advantages in this regard since it (i) has a low unit cost, (ii) can take advantage of locality, and (iii) can improve communications volumes. Procedures tested in two dimensional situations are being extended to three dimensions and preliminary performance results are encouraging.

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FINAL SCIENTIFIC REPORT

Air Force Office of Scientific Research Grant F49620-93-1-0218

Period:	1 April 93 through 31 March 1994
Title of Research:	Numerical Methods for Singularly Perturbed Differential Equations with Applications
Principal Investigator:	Joseph E. Flaherty

Department of Computer Science

Rensselaer Polytechnic Institute

Troy, New York 12180

ABSTRACT

During this one-year project, we continued our research on the development, analysis, and application of serial and parallel adaptive computational strategies for solving transient and steady partial differential systems. We concentrated on high-order methods and adaptive approaches that unite mesh refinement and coarsening (h-refinement), order variation (p-refinement), and mesh motion (r-refinement). Parallel computational techniques involved load-balancing and load-redistribution strategies for implementing these adaptive methods on distributed-memory MIMD computers. In particular, we have developed migration strategies that exchange finite elements between neighboring spatial domains of different processors. Effective load balancing in an adaptive setting requires speedy procedures since balancing must be performed frequently. Migration offers several advantages in this regard since it (i) has a low unit cost, (ii) can take advantage of locality, and (iii) can improve communications volumes. Procedures tested in two dimensional situations are being extended to three dimensions and preliminary performance results are encouraging.

We have initiated an effort to use our adaptive software to address problems associated with the manufacture of ceramic composites. During the year, we focused on modeling and solving problems involving "reactive vapor infiltration (RVI)." With RVI, the space surrounding a fibrous preform is filled with a powder that is infiltrated with a gas to react and form the matrix. The RVI process is faster than competing techniques (e.g., CVD, CVI) and offers the promise of reduced fabrication costs. Modeling, thus far, has involved mass and momentum considerations for a porous continuum. Volume expansion, as a result of chemical reaction, is included in the model to account for pores between powder grains closing, global volume expansion or contraction, and stress concentration or cracking. Results compare favorably with experiment and our computational investigation is motivating improved fabrication techniques.

1. Research Status and Key Results

During the past year we continued our investigations of adaptive techniques for (i) transient parabolic and hyperbolic partial differential systems and (ii) massively parallel computer systems. We also initiated a new effort to apply these computational strategies to problems arising in composite material fabrication. Highlights of our recent accomplishments include:

i. Developed the first adaptive hp-refinement technique for parabolic partial differential systems [3, 7, 13].[1]

ii. Developed efficient a posteriori error estimation techniques for parabolic [2, 3] and hyperbolic [6] systems.

iii. Developed load balancing procedures for adaptive computation on distributed-memory parallel computers [4-6, 8].

iv. Developed high-order methods for hyperbolic [5, 6] and singularly-perturbed parabolic systems [10].

v. Developed mathematical models for ceramic composite fabrication by a reactive vapor infiltration process [9].

vi. Applied adaptive techniques to compressible flow [11] and materials fabrication [9] problems.

Detailed descriptions of these endeavors. Lists of external interactions and publications appear in Sections 2 and 3, respectively.

1.1. hp-Refinement and Error Estimation

Adaptive method of lines (MOL) procedures utilize the power of sophisticated ordinary differential equations software and avoid developing special techniques for time integration. Using backward difference software and our variant of the singly implicit Runge-Kutta (SIRK) method [3, 13], we developed the first adaptive hp-refinement finite element strategy for parabolic systems [3, 7, 13]. Estimates of spatial discretization errors, obtained using p-refinement to create and solve local problems having forcing proportional to elemental or edge residuals of solutions, are efficient and converge to the actual finite element spatial errors at the correct rate [2].[2] The temporal variation of spatial errors may be neglected; hence, error estimates may be determined by solving local elliptic instead of

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[1] Numbers in brackets correspond to publications and manuscripts in Section 3.

[2] A two-dimensional analysis is being described in a manuscript under preparation "A Posteriori Error Estimation for the Finite Element Method of Lines Solution of Parabolic Systems," by S. Adjerid, I. Babuska, and J.E. Flaherty.

local parabolic problems. In typical applications, this reduces the time required to obtain an error estimate by approximately thirty percent.

With a goal of developing adaptive local refinement methods (LRMs), where local h- and p-refinement occurs in both space and time, we developed a variant of the SIRK method that offers several advantages when used with hp-refinement [3, 13]. SIRKs have high-stage order, A- and L-stability, efficiencies close to those of backward difference software, embedded error estimates, and a locality that is suitable for parallel computation. Temporal error estimates of each SIRK stage may be computed for the cost of one additional function evaluation [3]; thus, results at partial time steps may be accepted [13].

Spatial error estimates obtained by p-refinement [2, 7] have a limited range of applicability when applied to singularly-perturbed systems. Furthermore, solutions obtained by Galerkin techniques produce spurious oscillations when the computational mesh is too coarse within boundary and interior layers. Thus, without an a priori knowledge of layer locations, an adaptive system would produce a preliminary coarse-mesh solution containing spurious large-amplitude oscillations. Even with restricted range, an error estimator would (correctly) indicate enrichment in oscillatory regions. A correct solution could possibly be obtained through this enrichment; however, it would hardly be optimal since an excessively fine mesh would be needed to eliminate anomalous oscillations. Traditional approaches for overcoming this difficulty add artificial viscosity to the solution, thus, widening boundary layers to the current mesh width. These techniques are mostly successful with low-order methods for convection-diffusion systems; hence, they would require substantial modification to a high-order adaptive software system.

Using singular-perturbation theory, Adjerid et al. [10] developed a strategy for stabilizing solutions of singular-perturbation problems by changing the quadrature rule used to evaluate inner products. Among other things, they show that Radau and Lobatto quadrature, respectively, produce stable solutions of convection- and reaction-diffusion problems. These minor modifications to an adaptive software system confine large errors to layers regardless of whether or not their locations are known. The framework, furthermore, indicates how to develop quadrature rules for other singular perturbations.

Biswas et al. [6] and Devine et al. [5] developed superior solution limiting and error estimation schemes for a local finite element technique of Cockburn and Shu.[3] When used to solve hyperbolic systems of conservation laws, the new limiting strategies preserve an essentially non-oscillatory behavior near discontinuities while maintaining a high order of

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[3] B. Cockburn and C.-W. Shu, TVB Runge-Kutta Local Projection Discontinuous Finite Element Method for Conservation Laws II: General Framework, *Maths. Comp.*, **52** (1989), 411-435.

B. Cockburn and C.-W. Shu, TVB Runge-Kutta Local Projection Discontinuous Finite Element Method for Conservation Laws III: One-Dimensional Systems, *J. Comput. Phys.*, **84** (1989), 90-113.

accuracy in smooth regions. Following the ideas of Adjerid et al. [10], spatial error estimates utilize a p-refinement approach with superconvergence at Radau points ω compute efficient and (apparently) asymptotically correct results [5, 6].

1.2. Parallel Procedures

Our recent research on load balancing schemes for adaptive methods is intended for use with distributed-memory MIMD computation. An effective load balancing strategy should minimize the total execution time including the computational time, the communications time, and the time spent on reassigning processors to different spatial regions as a result of adaptivity. Thus, existing strategies that minimize communication on a fixed mesh will be sub-optimal because of their high per-step expense. Processor scheduling and load balancing must be dynamic and have a low unit cost and high parallel performance. Thus, we utilize incremental migration techniques where finite elements are interchanged between processors as the solution evolves and processor demands change.

Tiling [5, 6] is a modification of a global load balancing technique of Leiss and Reddy[4] who balanced local loading within overlapping processor neighborhoods. However, rather than using hardware interconnections, tiling neighborhoods are defined as a processor at the center of a circle of some predefined radius, all other processors within this circle, and all processors having finite elements that are neighbors of elements in the central processor. Each processor is the center of a neighborhood and loading is balanced within neighborhoods using local performance measurements. Work is migrated from a processor to any other processor within the same neighborhood. A priority system favors migration of elements having neighbors in the importing processor. We illustrate some performance characteristics through an example.

Example 1. We solved a linear advection equation on a two-dimensional rectangular domain on a 1024-processor nCUBE/2 hypercube at Sandia National Laboratories using the local finite element method [5, 6] with fixed-order (two) and adaptive p-refinement. The mesh for the fixed-order computation was selected so that its solution and the adaptive solutions had approximately the same error; however, the resulting error of the fixed-order solution was 66% greater than that of the adaptive solutions (cf. Table 1). The solution of the advection equation involved a relatively steep wave front that propagated across the domain at an oblique angle; thus, processor load balancing was necessary throughout the course of the computation.

Performance statistics, presented in Table 1, indicate that adaptivity reduced the total execution of the fixed-order method by 68% with a 23% loss in scaled parallel speed up

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[4] E. Leiss and H. Reddy, Distributed Load Balancing: Design and Performance Analysis, W.M. Keck Research Computation Laboratory, 5 (1989), 205-270.

	Fixed-Order	Adaptive p-Refinement	
	Without Tiling	Without Tiling	With Tiling
Global L^1 Error	6.41×10^{-2}	3.85×10^{-2}	3.85×10^{-2}
Total Execution Time (sec.)	1602	991	518
Avg. Work/Max. Work	0.855	0.359	0.896
Avg. Communication Time (sec.)	59	60	66
Max. Balancing Time (sec.)	0	0	25
Parallel Efficiency (%)	83	34	64

Table 1. Performance comparison for Example 1 using fixed-order (two) and adaptive p-refinement methods with and without balancing through tiling at each time step.

(parallel efficiency). Tiling, not necessary with fixed uniform-mesh computation, is clearly needed with adaptive techniques. Adaptive computational results with tiling are 48% faster than those without tiling and have an 88% better parallel efficiency.

While these results are encouraging, there are several ways of improving the tiling and migration strategies. Tiling, for example, has a slow distribution of balancing errors, it currently provides no improvement in communications volume, and it is constrained to two dimensions. DeCougny et al. [14] assemble the neighborhood workload requests of the tiling procedure [5, 6] into a forest of trees. Migration strategies may either occur between pairs of processors or by balancing the trees. These techniques are being implemented in three dimensions and are being interfaced to the SCOREC (Scientific Computation Research Center) mesh database. This mesh database controls mesh generation, adaptive h-refinement, and domain decomposition. Operators have been written for the rapid migration of elements between processor partitions and for updating information across partition boundaries as a result of h-refinement.

Example 2. The flow space surrounding half of a tandem helicopter has been discretized into a mesh of 94,069 tetrahedral elements. The projection of the mesh onto a plane is shown in Figure 2. Elements in this plane have been colored according to their partitioning on a 16-processor IBM SP-1 computer. Partitioning was done using inertial

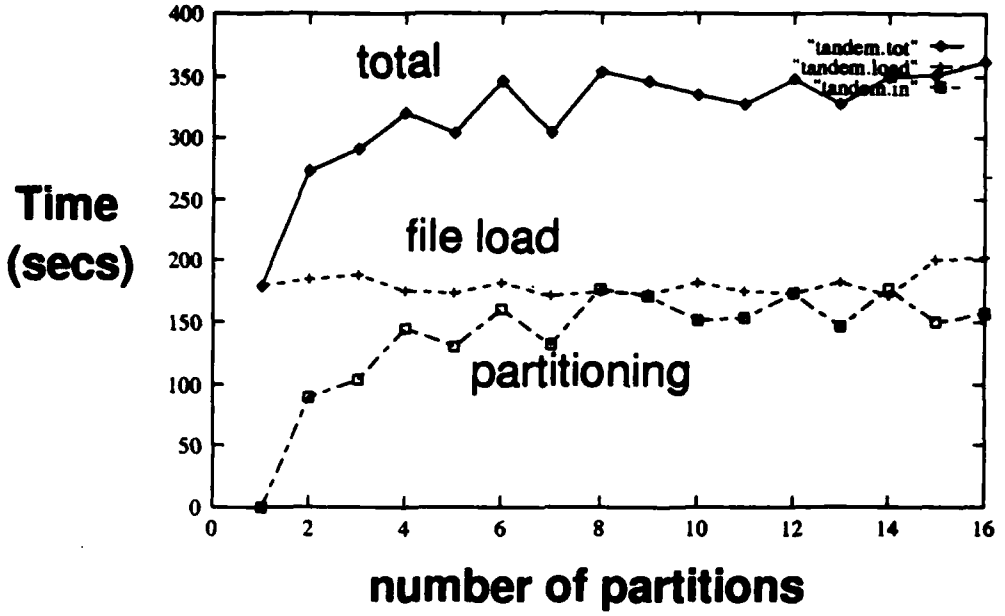

Figure 1. Computational time as a function of the number of partitions for the tandem helicopter mesh of Example 2.

recursive bisection (IRB) with migration. The IRB algorithm performs recursive bisection of the volume of the partition in planes parallel to the principal axes of inertia rather than along coordinate directions. IRB may provide an effective low-cost way of controlling communications cost in an adaptive setting. Elemental "masses" may be assigned in proportion to local polynomial degree to use the procedure with adaptive hp-refinement.

Computational effort is shown as a function of the number of partitions in Figure 2. The parallel complexity of IRB-migration strategy is linear in the number of elements N_Δ and, hence, has a constant time for a fixed mesh. This is verified by the results shown in Figure 2. The cost of maintaining the data structure during migration behaves similarly; thus, the total parallel computational cost is independent of the number of processors. Hence, the partitioning and redistribution algorithms have (near) perfect scalability.

Mesh generation underlying the SCOREC mesh database is done by an octree decomposition of physical space. This can be exploited to construct partitioning and load balancing algorithms that are faster than those for arbitrary unstructured meshes. Indeed, parallel partitioning can be performed by a two-step procedure that (i) determines cost metrics for all subtrees and (ii) partitions the octree according to these metrics [8, 14]. The partitions consist of octants that are each the root of a subtree and are determined by

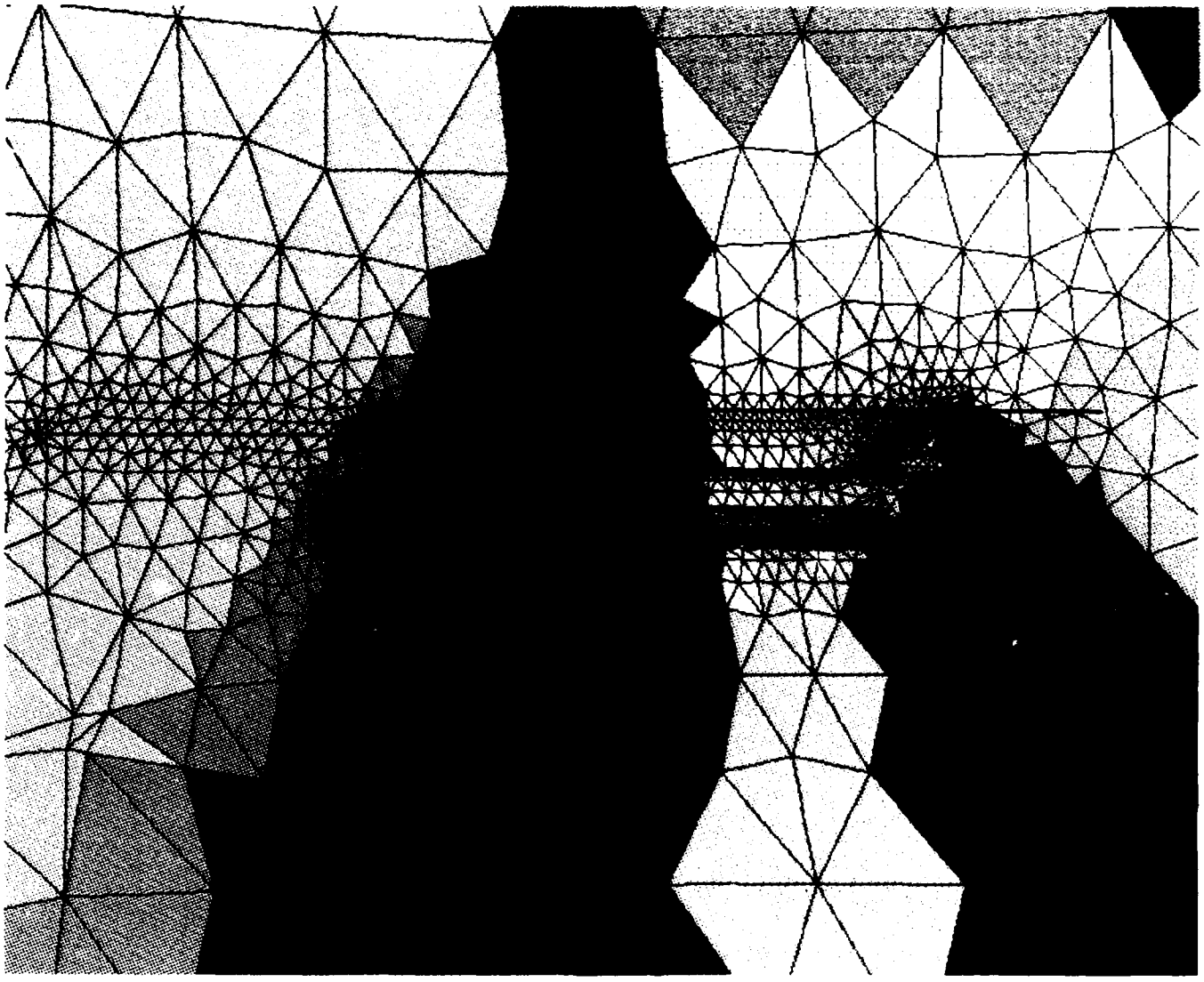

Figure 2. Tandem helicopter and the projection of a portion of the mesh surrounding it onto a plane. Colors indicate processor assignments using IRB with migration (cf. Example 2).

a truncated depth-first search.

Example 3. The tree-based partitioning strategy has been applied to a series of six finite-octree generated meshes [14] involving flow solutions of the Euler equations about airplanes, helicopters (cf. Example 2), wings, and cones. The number of elements in each mesh ranged from 16,000 to 293,000 with and average of 174,000. Partition quality is measured as the percent of element faces on inter-partition boundaries relative to the total number of faces of the mesh. In Figure 3, we display these percentages as a function of the optimal partition size. The variance between partition costs is as small as the cost of a leaf octant in all cases.

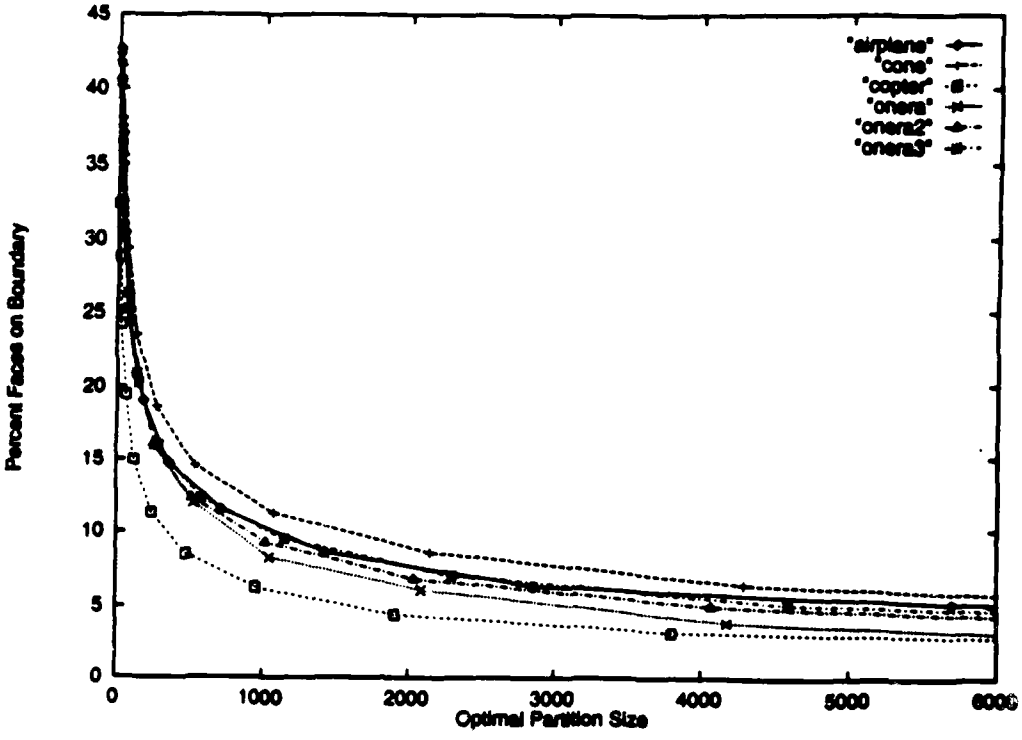

Figure 3. Performance of the octree partitioning algorithm on six meshes.

The performance metric shown in Figure 3 is a measure of the total surface area that partitions have in common. Smaller values require less communication relative to the local volume of data. The interface proportion is less than 12% when the partition size exceeds 1000 elements and drops to below 9% for partitions of 2000 elements or more.

This performance is comparable to recursive spectral bisection but has $O(N_\Delta)$ instead of $O(N_\Delta^2)$ complexity.[5]

We use the local finite element procedure [5, 6] with adaptive h-refinement and the octree partitioning technique to investigate the supersonic (Mach two) flow over a cone. Results, shown in Figure 4, follow several steps of h-refinement obtained by adding and deleting octants to and from the tree structure and generating a new mesh locally. Computations were performed on a Thinking Machines CM-5 computer at the University of Minnesota. The shock surface and pressure contours are shown in the upper portion of Figure 4 while planar projections of partitionings for 16- and 32-processor computers are shown below. Color on partition diagrams denotes membership in a partition. Use of the octree to balance processor loading is simple and fast. It offers a great deal of promise for both adaptive and standard computation.

1.3. Composite Materials Manufacturing

We have been working with Rensselaer Materials Scientists William Hillig and John Hudson to study fabrication problems associated with ceramic composites with a goal of manufacturing high-temperature composites more rapidly and less expensively than currently possible. Hillig and colleagues[6] describe a reactive vapor infiltration (RVI) process for manufacturing fiber-reinforced ceramic composites where silicon carbide (SiC) or alumina (Al_2O_3) fibers are mixed with molybdenum (Mo) powder and pressed at room temperature to form a porous preform. The preform is exposed to a silicon tetra-chloride ($SiCl_4$) and hydrogen (H_2) flow where molecular-surface reactions liberate Si which, when absorbed into the preform, reacts with Mo to form a molybdenum di-silicide ($MoSi_2$) matrix. Based on considerations of mass conservation, we developed the following model of the diffusion of Si into a porous pellet to form an intermediate (Mo_5Si_3) silicide layer and the desired $MoSi_2$ matrix:

$$\frac{D(\rho Y_i)}{Dt} := \frac{\partial(\rho Y_i)}{\partial t} + \mathbf{v}\cdot\nabla(\rho Y_i) = \nabla\cdot D_i \nabla(\rho Y_i) - \rho Y_i \nabla\cdot\mathbf{v} + \dot{r}_i, \quad i = 1, 2, \cdots, 4. \qquad (1)$$

The symbol ρ denotes the mixture density and $\mathbf{v}$ denotes the mixture velocity. The mass fraction, diffusivity, and production rate of species i are, respectively, denoted as Y_i, D_i, and $\dot{r}_i$, with index $i = 1, 2, 3, 4$, identifying Si, $MoSi_2$, Mo_5Si_3, and Mo. Reaction rates are much faster than diffusion rates and are described by Adjerid et al. [9].

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[5] A. Pothen, H. Simon, and K.-P. Liou, "Partitioning Sparse Matrices with Eigenvectors of Graphs, *SIAM J. Matrix Anal. Applics.* **11** (1990), 430-452.

[6] N. Patibandla and W.B. Hillig, "Processing of Molybdenum Di-silicide Using a New Reactive Vapor Infiltration Technique, *J. Am. Ceram. Soc.*," **76** (1993), 1630-1634.

With an initial *Mo* powder compressed to a porosity of 45%, the siliciding reactions are accompanied by a 158% volume increase that fills the pores between grains of *Mo* powder, but is excessive and may cause cracking. By a combination of analysis and experimentation, we discovered that material expansion and production times can be reduced by initiating the process with a mixture of powdered *Mo* and $MoSi_2$. Local and total volume expansion is modeled by approximating the reacting medium as a viscous fluid and coupling the viscous momentum equations to the mass conservation system describing reaction and diffusion. Thus,

$$\rho \frac{D\mathbf{v}}{Dt} + \mathbf{v}[\partial_t \rho + \nabla\cdot(\rho\mathbf{v})] = \nabla\cdot\mathbf{T} \tag{2a}$$

where the traction matrix $\mathbf{T}$ has components

$$T_{i,j} = (-p + \lambda\nabla\cdot\mathbf{v})\delta_{i,j} + \mu(\partial_{x_j} v_i + \partial_{x_i} v_i) \tag{2b}$$

with λ and μ being Lamé parameters, p being the pressure, and $\delta_{i,j}$ being the Kronecker delta.

A volumetric flow relation describes the behavior of the pores between powdered grains and we derive this as

$$-\frac{D\nu}{Dt} + (\nabla\cdot\mathbf{v})(1-\nu) = \sum_{i=1}^{4} \frac{1}{\hat{\rho}_i}[\dot{r}_i + \nabla\cdot D_i \nabla\rho Y_i], \tag{3}$$

where ν is the porosity and $\hat{\rho}_i$, $i = 1, 2, 3, 4$, are the species densities.

Example 4. We solve (1-3) using our adaptive finite element MOL software and report some results obtained by Adjerid et al. [9]. Specific parameter values are given therein and, for the most part, are not duplicated here. We compare computed and observed results for the square of the thickness of the $MoSi_2$ layer as a function of time for three temperatures in the left portion of Figure 5. Computed and experimental results are in excellent agreement with deviations being less than 10%.

Solutions were obtained by hpr-refinement and, in Figure 6, we show the spatial mesh used at four different times. Colors correspond to concentrations of $MoSi_2$ with red denoting a high concentration and blue denoting a low concentration. The transition from red to blue corresponds to the formation of Mo_5Si_3 in a narrow layer between the fabricated $MoSi_2$ (red) and the unreacted *Mo* (blue). A coarse mesh and first-order method are used away from the reaction zone while finer meshes and high-order methods are used near the reaction. The mesh is concentrated near the reaction zone and is moving to account for the expansion as the reaction occurs.

Results shown on the right of Figure 5 compare computed and observed values of the square of the thickness of the $MoSi_2$ layer as a function of time for three initial concentrations. As noted, fabrication times can be shortened by initiating the reaction with a

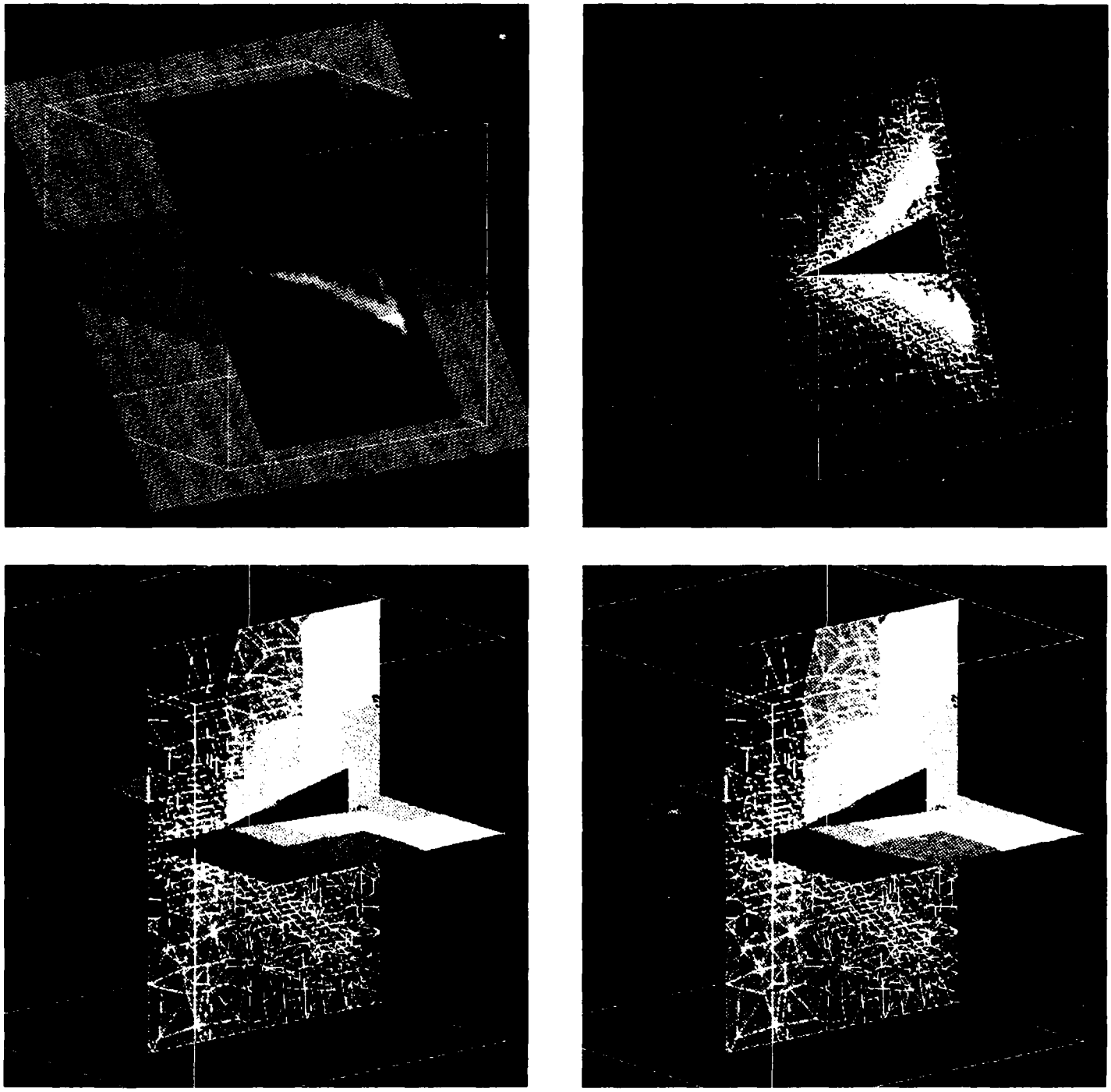

Figure 4. Shock surface and pressure contours found when computing the Mach two flow past a cone having a half angle of 10° (top). Partitions of the mesh into 16 (lower left) and 32 (lower right) pieces.

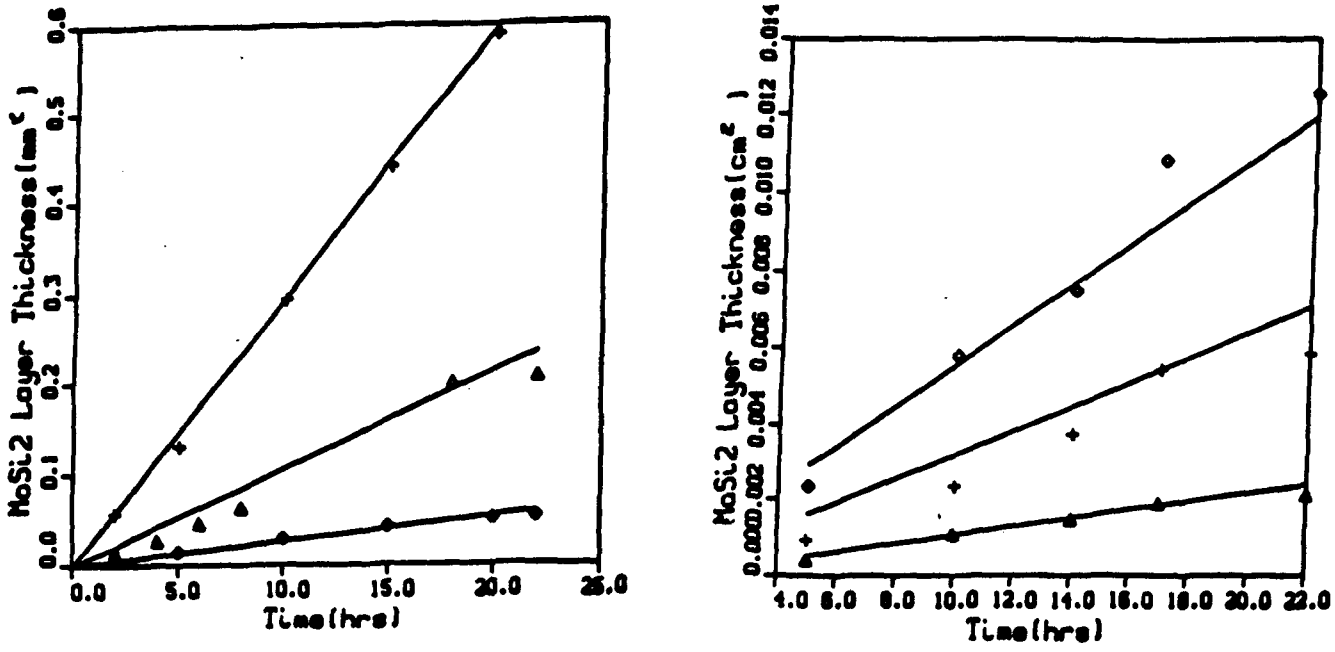

Figure 5. Comparison of computed and observed values of the square of the thickness of the $MoSi_2$ layer as a function of time. On the left, comparisons for temperatures of 1100 °C (diamonds), 1200 °C (triangles), and 1300 °C (plusses). On the right, comparisons for initial concentrations of all *Mo* (triangles), 50% *Mo* and 50% $MoSi_2$ (plusses), and 30% *Mo* and 70% $MoSi_2$ (diamonds).

mixture of *Mo* and $MoSi_2$. The particular initial concentration of half *Mo* and half $MoSi_2$, compressed to 45% porosity, fills pores to within 8% with (essentially) no volume expansion. Our model predicted both the correct growth rate of the silicide layer the lack of volume expansion.

Finally, in Figure 7, we exhibit the thickness of the $MoSi_2$ layer at four times for a two-dimensional computation using adaptive h-refinement. Once again, high concentrations are colored red and low ones are colored blue. The mesh used for the computation at the indicated time has been superimposed. The bottom and right edges of each figure are lines of symmetry. The silicide layers progress inward from the top and left edges to build the $MOSi_2$ and Mo_5Si_3 layers. Reaction fronts are sharp and the mesh is concentrated in and following these regions. The Mo_5Si_3 layer is a narrow zone that precedes the $MoSi_2$ layer into the unreacted molybdenum.

Work on materials processing is just beginning and is the focus of our current grant with AFOSR (F49620-94-1-0200). Our intentions are to

i. Improve the mechanical coupling model by including elastic and plastic deformation.

ii. Develop microstructure models by studying the flow around grains and fibers and through pores.

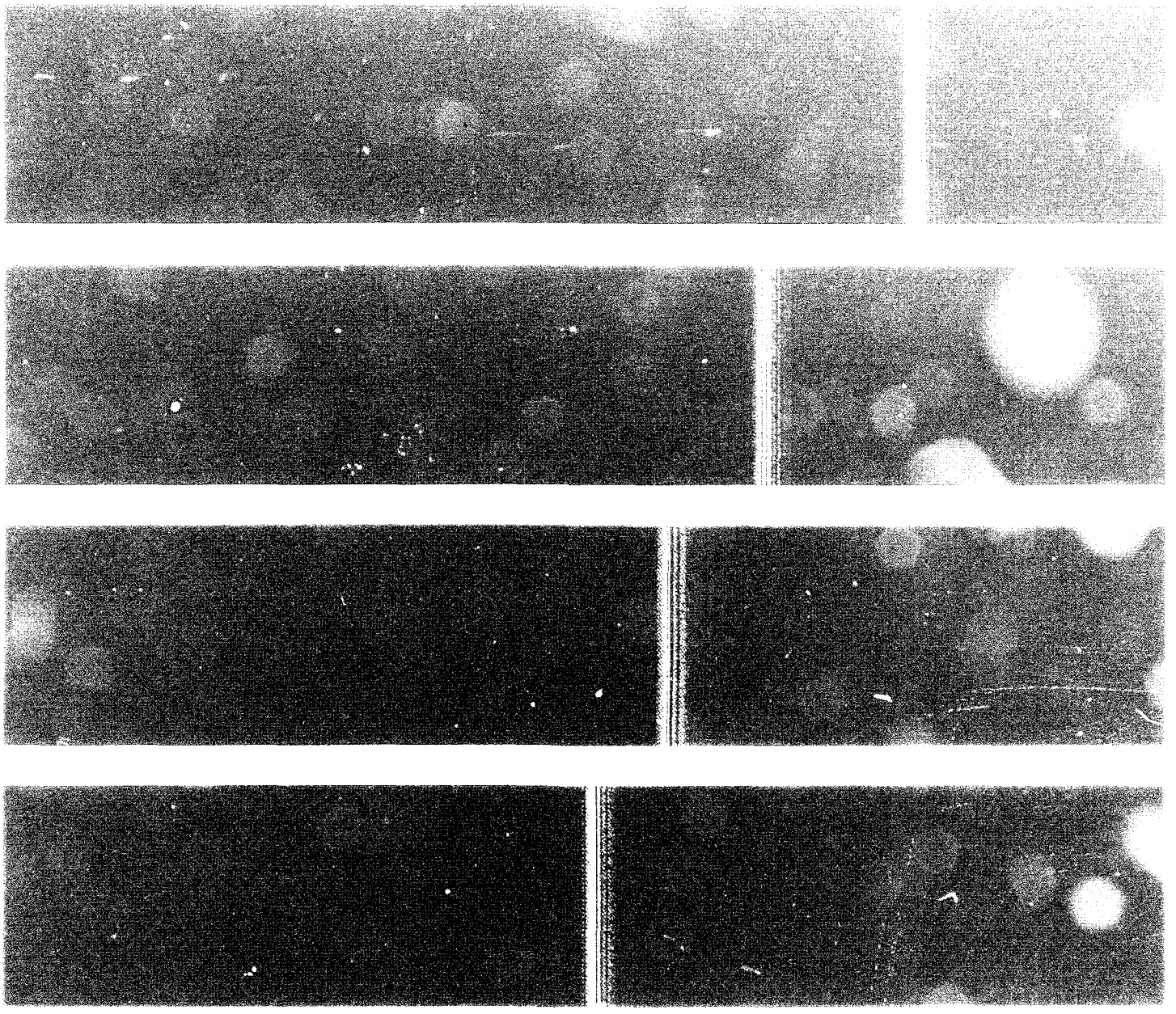

Figure 6. Mesh and chemical concentrations at four times. Colors correspond to concentrations of $MoSi_2$ with red denoting a high concentration and blue denoting a low concentration.

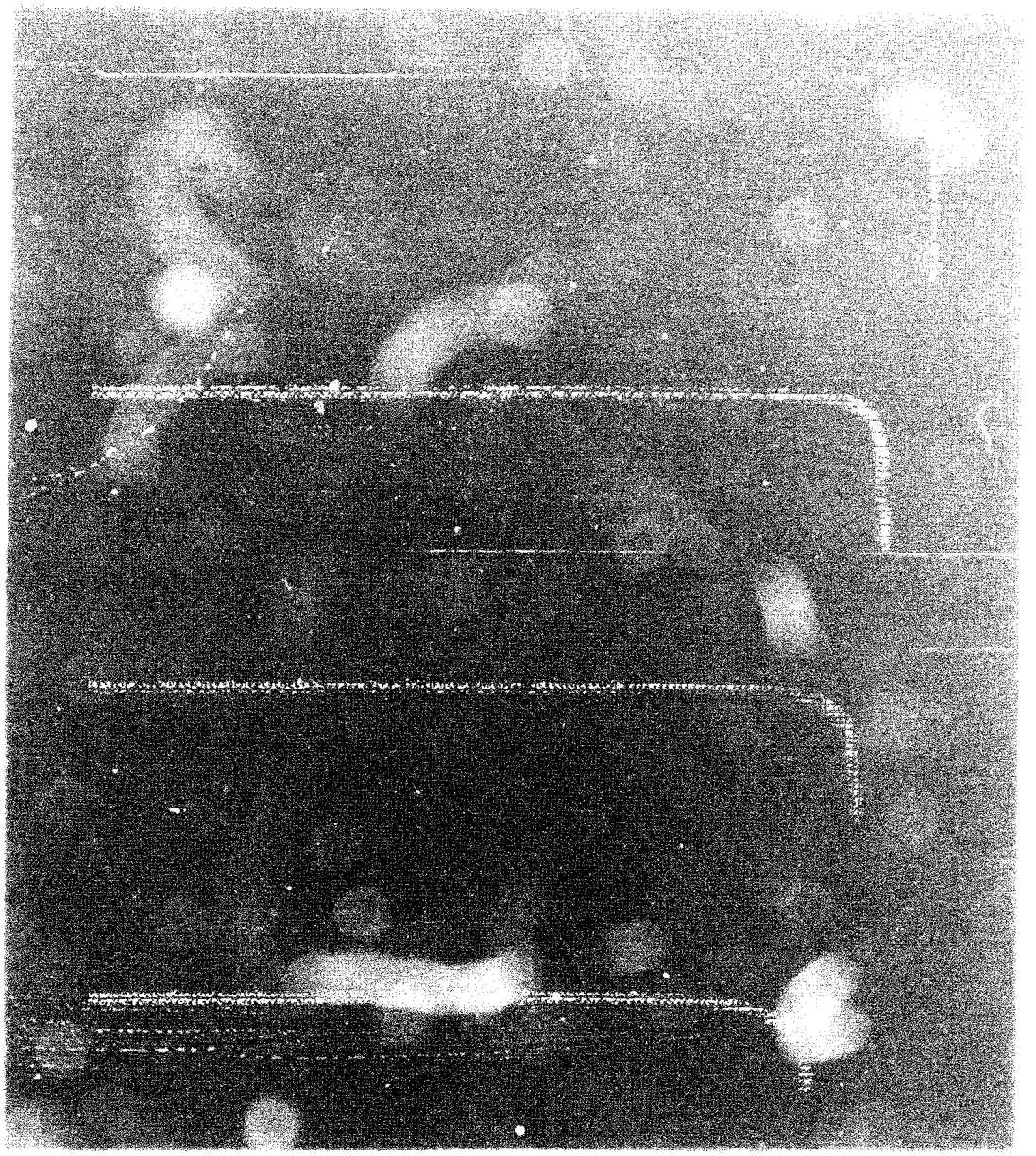

Figure 7. Concentration of $MoSi_2$ at 40 minutes, 3.5 hours, 7.5 hours and 11.5 hours (top to bottom) of reaction. High concentrations are shown in red and low concentrations in blue.

iii. In collaboration with Wright Laboratories and Centric Engineering, study fiber coating processes.

iv. Study the effects of grain shape, initial porosity, and initial composition in an effort to further reduce processing times.

The software and model are not limited to RVI and we will investigate its use with other fabrication processes.

2. Interactions

Invited lectures and other interactions by personnel involved with research supported by this grant follow.

1. Flaherty lectured on "Adaptive and Parallel High-Order Methods for Conservation Laws," University of Kentucky, Lexington, April 21, 1993.

2. Flaherty attended the *AFOSR Contractor/Grantee meeting* at Washington University, May 20-21, 1993. He lectured on "Adaptive and Parallel Computational Techniques for Partial Differential Equations."

3. Flaherty attended the *AFOSR MCAD Review* at the University of Michigan, May 27, 1993. He presented an evaluation of the material to N. Glassman and C. Holland of AFOSR.

4. Marsha Berger of New York University visited Flaherty and Mark Shephard, of Rensselaer's Scientific Computation Research Center on June 16, 1993, to discuss adaptive techniques on Cartesian grids.

5. Flaherty, with Julian Cole and Donald Drew of Rensselaer and Bernard Matkowsky of Northwestern University and Lu Ting of New York University, organized and ran a workshop on *Perturbation Methods in Physical Mathematics* at Rensselaer, June 23-26, 1993. The workshop honored Joseph B. Keller and recognized his seventieth birthday. More than one hundred scientists attended the symposium. Flaherty lectured on "Adaptive Numerical Procedures for Singularly Perturbed Partial Differential Equations." Written proceedings of this workshop will be published as a special issue of *SIAM Journal of Applied Mathematics.*

6. Flaherty visited N. Glassman, C. Holland, and M.Q. Jacobs at AFOSR headquarters to discuss the use of adaptive methods for addressing materials processing problems involving ceramic composites.

7. Flaherty, with Ivo Babuska of the University of Maryland, William Henshaw of IBM, John Hoppcroft of Cornell University, Joseph Oliger of Stanford University, and Tayfun Tezduyar of the University of Minnesota organized a summer program on *Modeling, Mesh Generation, and Adaptive Numerical Methods for Partial Differential Equations* at the Institute for Mathematics and its Applications of the University of Minnesota, Minneapolis, July 6-23, 1993. Flaherty lectured on "Adaptive and Parallel High-Order Methods for Conservation Laws."

8. Flaherty attended the *AFOSR Working Group Meeting on Modeling and Design for High Pressure Crystal Growth Processes* at Rome Laboratories, Hanscom AFB, July 29-30, 1993.

9. Flaherty lectured on "Computers: Small and Large, Serial and Parallel," "An Introduction to the Parallel Solution of Finite Element Problems," "Parallel Solution Procedures on Shared and Distributed Memory Computers," and "Adaptive and Parallel Finite Element Methods for Conservation Laws," Summer workshop on *Frontiers in Finite Element Technology,* Győr, August 16-19, 1993. This Hungarian-sponsored symposium was attended by faculty and graduate students from former iron-curtain countries. The three keynote lecturers. Flaherty, Ivo Babuska of the University of Maryland, and Barna Szabo of Washington University, each presented a sequence of four talks.

10 Flaherty lectured on "Adaptive and Parallel High-Order Methods for Conservation Laws" at the University of Toronto, February 25, 1994.

11. Flaherty and John Hudson of Rensselaer's Materials Engineering Department were to fly to Wright Laboratories to meet with James Malas and Stephen LeClair on March 4, 1994 to discuss mutual interest in ceramic fiber coating processes. Poor weather forced the cancellation of their private flight, but Hudson attended the meeting via commercial transportation. There meeting resulted in the submission of an STTR proposal on fiber coating processes with Rensselaer and Centric Engineering as academic and industrial partners.

12. Flaherty attended the *Workshop on Basic Phenomena in Plasticity* at Carnegie Mellon University, March 17-19, 1994. He lectured on "Adaptive Methods for Partial Differential Equations with Application to Shear Band Formation."

13. Flaherty presented a lecture on "Adaptive Methods for Parabolic Partial Differential Equations" at the University of Buffalo, May 5, 1994.

3. Publications and Manuscripts

Papers published or in press and submitted manuscripts on research supported by this contract follow.

Publications

1. D.C. Arney, R. Biswas, and J.E. Flaherty, "An Adaptive Mesh Moving and Refinement Procedure for One-Dimensional Conservation Laws," *Appl. Numer. Math.*, **11** (1993), 259-282.

2. S. Adjerid, J.E. Flaherty, and Y.J. Wang, "A Posteriori Error Estimation with Finite Element Methods of Lines for One-Dimensional Parabolic Systems," *Numer. Math.*, **65** (1993), 1-21.

3. P.K. Moore and J.E. Flaherty, "High-Order Adaptive Finite Element-Singly Implicit Runge-Kutta Methods for Parabolic Differential Equations," *BIT*, **33** (1993), 309-331.

4. M. Benantar, J.E. Flaherty, C. Ozturan, M.S. Shephard, and B.K. Szymanski, "Parallel Computation in Adaptive Finite Element Analysis," Chap. 7 in C.A. Brebbia and M.H. Aliabadi, Eds., *Adaptive Finite Element and Boundary Element Methods*, Computational Mechanics Publics., Southampton, co-published with Elsevier Applied Science, London, 1993.

5. K. Devine, J.E. Flaherty, S. Wheat, and A.B. Maccabe, "A Massively Parallel Adaptive Finite Element Method with Dynamic Load Balancing," Tech. Rep. SAND93-0936C, Sandia National Laboratories, 1993. Also, Proc. Supercomputing '93, to appear.

6. R. Biswas, K.D. Devine, and J.E. Flaherty, "Parallel Adaptive Finite Element Methods for Conservation Laws," *Appl. Numer. Maths.*, **14** (1994), 255-284.

In Press

7. S. Adjerid, J.E. Flaherty, and Y.J. Wang, "Adaptive Method-of-Lines Techniques for Parabolic Systems," Proc. *First International Symposium on Method of Lines, Surfaces and Dimensional Reduction in Computational Mathematics and Mechanics*, Athens, 1994, to appear.

8. R. Loy and J.E. Flaherty, "Parallel Adaptive Load-Balancing Schemes for Three-Dimensional Conservation Laws," to appear in W. Ames, Ed., *Proc. IMACS World Congress*, 1994, Atlanta.

9. S. Adjerid, J.E. Flaherty, W. Hillig, J. Hudson, and M.S. Shephard, "Adaptive Method of Lines Techniques for Vapor Infiltration Problems," to appear in W. Ames,

Ed., *Proc IMACS World Congress,* 1994, Atlanta.

10. S. Adjerid, M. Aiffa, and J.E. Flaherty, "High-Order Finite Element Methods for Singularly-Perturbed Elliptic and Parabolic Problems," *SIAM J. Appl. Math.,* 1994, to appear.

11. B.E. Webster, M.S. Shephard, Z. Rusak and J. E. Flaherty, "An Automated Adaptive Time-Discontinuous Finite Element Method for Unsteady Compressible Airfoil Aerodynamics," *AIAA Journal,* 1994, to appear.

Submitted Manuscripts

12. M. Benantar, J.E. Flaherty, and M.S. Krishnamoorthy, "Triangle Graphs," 1992, submitted for publication.

13. J.E. Flaherty and P.K. Moore, "An hp-Adaptive Method in Space and Time for Parabolic Systems," Tech. Rep. 93-15, Dept. Comp. Sci., Rensselaer Polytechnic Institute, June 1993. Also, submitted for publication.

14. H.L. deCougny, K.D. Devine, J.E. Flaherty, R.M. Loy, C. Ozturan, and M.S. Shephard, "Load Balancing for the Parallel Adaptive Solution of Partial Differential Equations," Tech. Rep. 94-8, Dept. Comp. Sci., Rensselaer Polytechnic Institute, December 1994. Also submitted for publication.